Synergy, Sigmoids and the Seventh-Year Trifurcation

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SUMMARY

Whereas linear and exponential growth have become relatively familiar metaphors, second-order growth processes, as exemplified by autocatalytic chemical reactions, have not yet entered the vernacular. Derek J. de Solla Price has discussed sigmoid growth, but described it as a three-phase process. It is shown here that these three phases are more apparent than real, with initial slow growth, intermediate rapid growth and ultimate saturation all characteristic of the same second-order process. A model, arrived at by the superposition of several sigmoid curves, is proposed for the periodic critical decision points which occur during the course of a career or the life time of an organization. These decision points correspond to a trifurcation, leading to renewed sigmoidal growth, saturation, or a rapid phase-out.

Synergy, sigmoids and the seventh-year trifurcation (A visual metaphor)

I. INTRODUCTION

It was in the Spring of 1980, while teaching a Freshman seminar entitled "Can Renaissance Man Survive in a Competitive Culture?" at Harvard that I was reminded of Derek J. de Solla Price's sigmoid curves (Price, 1969).

The seminar, after concluding that Renaissance Man is a twentieth century concept, ephemeral and vaguely desirable, rather than a fifteenth or sixteenth century reality, addressed itself to the phenomenon of changes in styles, the process by which tastes change and echoes of earlier fashions reenter fashion consciousness. The slow initial growth of a style or fashion in its 'grass roots' stage, the rapid acceptance when the style enters public consciousness, and the levelling off when the style becomes established and 'traditional', are beautifully represented by the metaphor of the sigmoid curve, shown in Fig. 1.

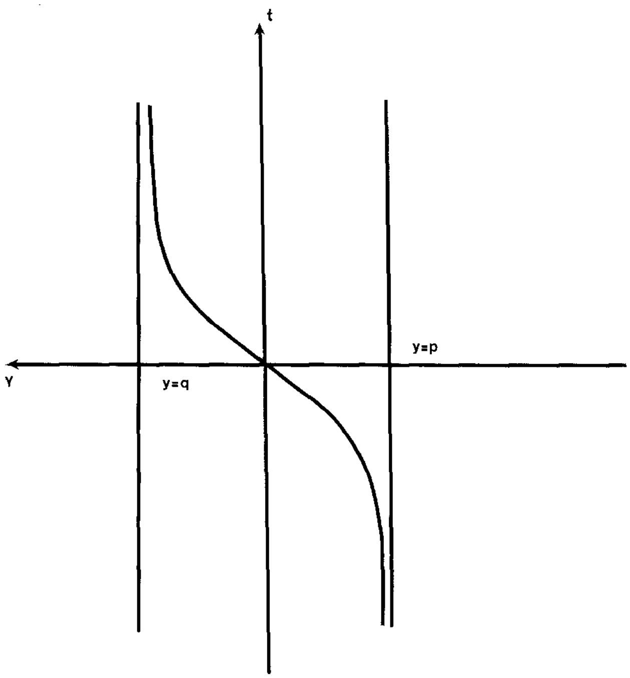

Fig. 1. Sigmoid growth curves.

A graph is a visual metaphor for a process. It may be a representation of large numbers of data, but it may also be used to conceptualize a process. In the nineteen sixties it became fashionable to observe that extrapolation of the rate of growth of the scientific population to the year 2000 A.D. would imply that by that year the number of scientists would exceed the number

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of humans on earth. Exponential growth became part of the vernacular even though at times implying the absurd.

In 1969 Price showed that a variety of processes, when studied over a sufficient time span (for instance, the rate at which European universities were founded during the Middle Ages and thereafter), followed the sigmoid curve of Fig. 1. Price referred to three distinct phases, two of slow growth, separated by one of very rapid growth.

Two questions arise with regard to sigmoidal growth. In the first place, one wonders whether the three phases observed by Price are in fact discrete; secondly, whether external changes or influences need to be postulated to explain the transitions from slow to fast to slow growth. These questions will be resolved below.

The Freshman seminar concluded by dismissing the concept of Renaissance Man, and instead defined the Synergetic Person as follows:

(1) This person has a variety of interests, and

(2) possesses a certain minimal skill in each,

(3) so that their combination in one and the same person produces a result superior to that which would result if these interests and skills were distributed over several separate persons.

The consensus of the seminar participants was that such a Synergetic Person has an excellent survival chance, through lateral mobility, but is not very upwardly mobile, because the so-called ladder of success is neither very relevant nor very accessible to the Synergetic Person. After studying the sigmoidal process and resolving the questions posed above, we shall use its curves metaphorically, to analyze the career decisions which create the Synergetic Person and the mechanisms which produce or retard the production of Synergetic Persons.

II. GROWTH FUNCTIONS

Generally, the rate of change of a function $y(t)$ depends explicitly on the function itself as well as on time:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y, t)$$

In special cases, however, the growth rate does not explicitly depend on time. An example is the growth of capital as a result of continuously compounding interest, another is the disintegration of radioactive matter. In the first instance the growth is positive, in the second negative, but in each case the *relative* rate of growth is independent of time, so that the *absolute* rate depends on y only.

Generally, we call functions whose rate of growth is explicitly independent of time, *growth functions*. They satisfy the general differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y) \tag{1}$$

where $\partial f/\partial t = 0$.

When $f(y)$ is constant, y increases or decreases linearly with time; we call this type of growth processes zeroth-order processes, because $f(y)$ depends on the zeroth power of y. Similarly, first-order processes have a rate of change that depends linearly on y:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ay + b \tag{2}$$

The general solution of eqn. (2) is:

$$y = \left(y_0 + \frac{b}{a}\right)\mathrm{e}^{at} - \frac{b}{a} \tag{3}$$

where y_0 is the value of y at time $t = 0$.

When $b = 0$, we find the special case of constant relative growth, exemplified above by continuously compounded interest and radioactive disintegration, which are thus shown to be first-order growth processes.

III. SECOND-ORDER GROWTH FUNCTIONS

It is logical to consider second-order processes next; their growth rate is a function of the second power in y, and, of course, independent of t. In this case, the function $f(y)$ in eqn. (1) has two roots, p and q, so that $f(p) = 0$ and $f(q) = 0$. For cases of interest here, we assume that p and q are real numbers, $p < q$. Without loss of generality, we can write:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = c(y - p)(q - y) \tag{4}$$

This form of the second-order growth equation is chosen such that if $c > 0$, $\mathrm{d}y/\mathrm{d}t > 0$ in the range $p < y < q$. It follows from eqn. (4) that the growth rate is small when y is close in value to p or to q. Differentiation of both sides of eqn. (4) shows that maximum growth occurs when $y = \frac{1}{2}(p + q)$, i.e. half-way between p and q.

It is seen that this behavior is close to sigmoidal. To confirm this by solving eqn. (4), we make the following substitutions:

$$z = (2y - q - p)/(q - p)$$

$$x = \tfrac{1}{2}c(q - p)t,$$

so that eqn. (4) becomes:

$$\frac{dz}{dt} = (1+z)(1-z) \tag{5}$$

For the range $-1 < z < +1$, corresponding to $p < y < q$, eqn. (5) yields $z = \tanh x$, with the origin in x, hence in t, chosen such that $x = 0$ and $t = 0$ when the growth rate is maximum. Therefore,

$$y = \tfrac{1}{2}(p+q) + \tfrac{1}{2}(q-p)\tanh[\tfrac{1}{2}c(q-p)t] \tag{6}$$

This function corresponds exactly to the sigmoid function of Fig. 1: the asymptotes occur at $y = p$ and $y = q$, and maximum growth at $y = \frac{1}{2}(p+q)$.

We may therefore conclude that sigmoid growth is characteristic of second-order growth processes, much as exponential growth is of first-order growth. The saturation phenomenon occurs without external interference, but is inherent in the process. Furthermore, the three discrete phases are more apparent than real: there is a gradual increase in the growth rate toward a maximum, followed by a gradual asymptotic decrease.

Second-order growth processes are familiar in chemical kinetics, for instance when a process is catalysed by one of its own products, a so-called autocatalytic reaction. In such an instance, the initial phase of the process is very slow, until sufficient catalyst is formed, when the process speeds up dramatically; saturation sets in when the original reagent becomes used up. An example is the conversion of trypsinogen into trypsin, the latter catalysing the reaction (Frost and Pearson, 1961).

Could one, in the early stages, distinguish between first- and second-order growth processes? When $t \ll 0$, i.e. at times long before the time when maximum growth occurs in second-order processes:

$$\tanh[\tfrac{1}{2}c(q-p)t] = -1 + \exp[c(q-p)t];$$

accordingly, in the early stages second-order growth looks exponential, hence is indistinguishable from a first-order process.

We have seen that sigmoid growth is characterized by a period of relatively rapid growth between the initial and final periods of very slow growth. Although it was shown that there is no sharp transition between these phases, it is true that at least 99.8% of the change-over from p to q occurs over the period $-7\Delta t/2 < t < +7\Delta t/2$, where

$$\Delta t = 2/c(q-p) \tag{7}$$

The time interval Δt is characteristic of the process, analogous to a half-life in first-order processes. It is smallest when the asymptotic values of y are far apart, and when c, an indicator of the absolute growth rate, is also large.

IV. HYPO- AND HYPER-CRITICAL PHASES

In the previous section we solved eqn. (4) for the range $p < y < q$. To find the solution outside this range, we use the following peculiar property of eqn. (5).

If z is a solution of the equation $dz/dt = (1+z)(1-z)$, then $u = 1/z$ is a solution of the same equation. (This property is easily proven by direct substitution.) Using this property, we see that for $y > q$ and for $y < p$

$$y = \tfrac{1}{2}(p+q) + \tfrac{1}{2}(q-p)\coth[\tfrac{1}{2}c(q-p)t] \tag{8}$$

This function is shown in Fig. 2. It consists of two branches, one having $y > q$, the other $y < p$;

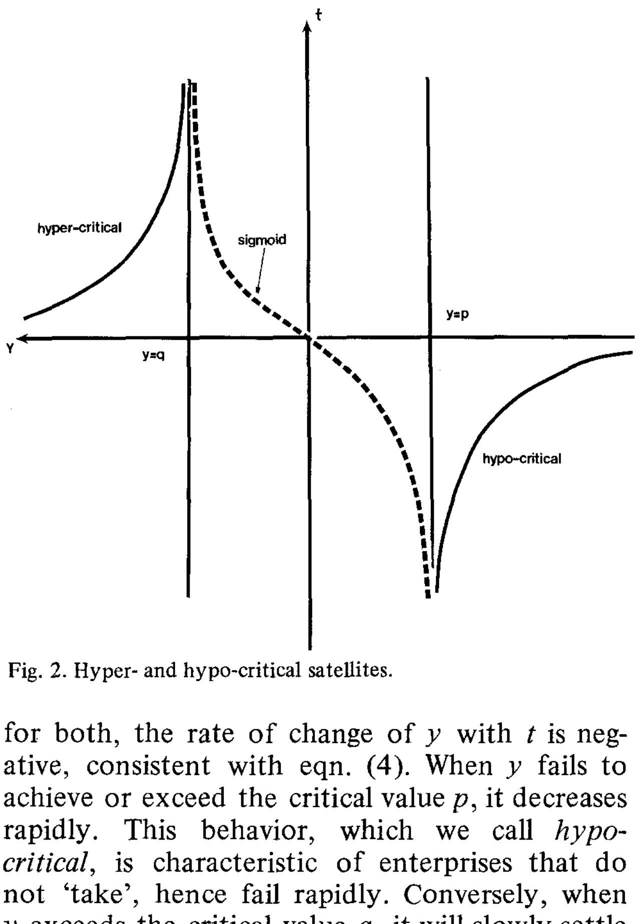

Fig. 2. Hyper- and hypo-critical satellites.

for both, the rate of change of y with t is negative, consistent with eqn. (4). When y fails to achieve or exceed the critical value p, it decreases rapidly. This behavior, which we call *hypo-critical*, is characteristic of enterprises that do not 'take', hence fail rapidly. Conversely, when y exceeds the critical value q, it will slowly settle down to that asymptotic limit. This behavior is characteristic of the over-qualified individual who is unable to perform to maximum capacity and eventually reaches a lower level of operation. It is called *hyper-critical*.

V. THE SEVENTH-YEAR TRIFURCATION

Let us now consider the following scenario; it is not a real case history, but a composite of various actual events. It is very likely that most readers recall similar experiences, for it is thought to be typical of common occurrences.

"The college town of Bradfield had for some years enjoyed special Christmas events, at which faculty, students and citizens of the town joined to provide instrumental musical support. A new member of the music faculty arrived, organizing the musicians into a chamber orchestra having regular weekly rehearsals and giving two public concerts each year. Eventually, the group becomes well-known, and is invited to travel to neighboring communities for additional performances. The conductor decides to apply for support from the National Endowment of the Arts in order to turn the ensemble into a regional orchestra. A crisis develops within the orchestra when some members, lacking the skill or time for such an ambitious undertaking, feel compelled to resign. However, the conductor prevails, and manages to attract some professionals from a city about fifty miles away to pull the new Bradfield Sinfonietta together. The grant application is successful, and the orchestra eventually makes some recordings in the acoustically perfect college chapel.

Professor Sattely, of the History Department, is one of those whose mastery of his instrument, the clarinet, was not such that could keep up with the professional demands of the new orchestra. Unfortunately, there were few other opportunities for performance, so he gradually got out of practice, and eventually sold his clarinet. His wife, who was an alto in the Bradfield Community Chorus, kept up with her singing, and eventually became president of that organization."

In this scenario, the informal Christmas performances may be viewed as the initial, slow-growth phase of the Bradfield chamber orchestra. The arrival of the new member of the music department signifies the rapid-growth phase. One might say: "Suppose that this conductor had not arrived?" Well, then there might not be a Bradfield Sinfonietta at all, or someone else might have come along. The fact is, however, that many organizations that do exist have grown out of a slow-growth 'grass roots' beginning. It might also have happened that after a number of Christmas parties the custom was not continued: the ensemble failed to achieve a critical value p, and rapidly phased out of existence.

What happened after the period of rapid growth is quite interesting, because it is very common, but usually catches people unawares. This is that the momentum, in this case injected by the new conductor, keeps the orchestra from settling into an 'establishment' amateur ensemble, as the Bradfield Community Chorus and Mrs. Sattely had done. The latter, having had their maximum growth phase, happily pursued the top of the sigmoid curve and settled into a venerable and much-appreciated tradition. The Bradfield Sinfonietta, on the other hand, began a new life and a new sigmoid curve. Professor Sattely, being faced with the fact that, unlike the Chorus, the orchestra was unable or unwilling to follow the same sigmoid curve along which it had climbed to relative success, found his own skill as a clarinettist 'below-p', and resigned from life as a musician.

The Sinfonietta, and Professor and Mrs. Sattely were used here to illustrate that within the space of a human career there are periodic critical points. The Biblical seven years, both as used in the parable of the fat and lean years (Pharaoh's dream) and in the command that the land should rest every seventh year and lie fallow, are examples of this rhythm. Universities grant sabbatical leaves on the basis of this same rhythm, and time their tenure decisions similarly. Marriage counselors are concerned about the 'seventh-year itch', and career counselors frequently advise a job change every seventh year.

To illustrate the seventh-year crisis, we have, in Fig. 3, superimposed portions of three sigmoid curves and their hypo- and hyper-critical satellites. We have chosen approximately one year for the characteristic time (cf. eqn. (7)), so that 99.8% of the change from lower to upper asymptote is achieved in about seven years. It is observed that after the period of slow growth followed by the rapid-growth phase, a trifurcation presents itself, and critical choices must be made. Like Mrs. Sattely and the Community Chorus, one might become venerable by staying on the asymptote of the old sigmoid. But like the Bradfield Sinfonietta, one might opt to enter a new sigmoidal phase. However, this involves a risk. If the NEA grant had not been received, the hypo-critical route would very likely have resulted, unless the group would have been satisfied to pursue the hyper-critical phase of the old sigmoid process back to the old asymptote.

As in Bradfield, we encounter the rhythmic cycle of slow growth—fast growth—trifurcation in our own lives. Some organizations and enterprises fail right away. Some are successful, and become traditions after a single cycle, others

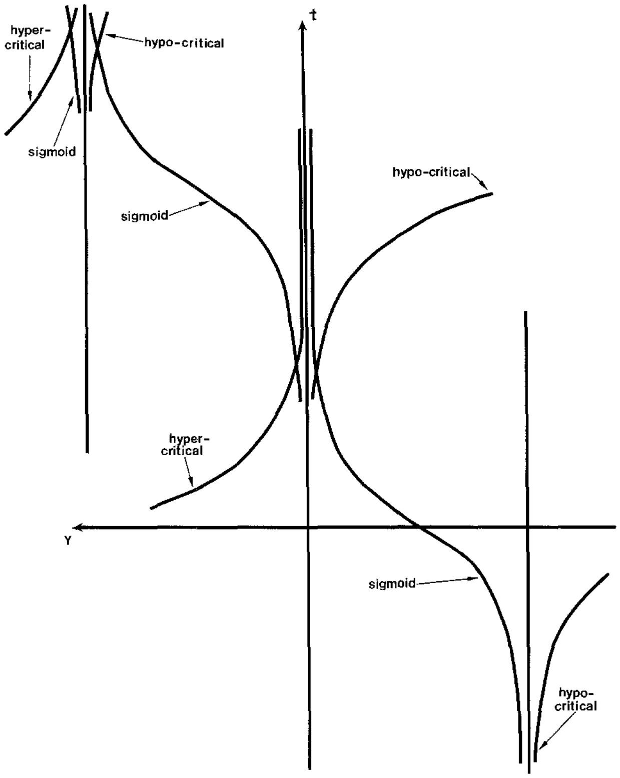

Fig. 3. Seventh-year trifurcation.

transform and enter a new cycle, and others, resisting becoming traditions, fail in their attempted transformation. Within organizations which have achieved a stable tradition, newcomers will, after an initial period of adjustment, rise to prominence, and following this either settle into becoming a prestigious fixture, or go on to greener pastures. Rarely does one see the same individual go through a second period of rapid growth within the same organization.

Synergetic Person's lateral moves are represented in Fig. 3 as jumps to new sigmoid curves every seventh year. S.P.'s career is characterized by periodic maxima in dy/dt, i.e. of maximum momentum. The specialist, on the other hand, is characterized by few, if any, transfers to new growth curves: the major push comes early, and fame and fortune reinforce the decision to stay on the asymptote of the same sigmoid curve which had produced these fruits in the first place.

VI. HIGHER-ORDER GROWTH FUNCTIONS

Because of the fundamental difference between first- and second-order processes, one might wonder whether higher-order growth would present yet further insights. Additional roots of $f(y)$ in eqn. (4) will produce additional asymptotes, between which growth would be alternately positive and negative. Trifurcations would occur as before, but the hypo- and hyper-critical satellites, instead of going to or coming from infinity, would look like inverted sigmoids. Only in the unbounded regions would the satellites still be able to become infinite. However, there do not appear to be significant differences between second- and higher-order growth processes relevant to our study.

VII. SUMMARY AND CONCLUSIONS

We have shown that sigmoidal processes are characteristic of second-order growth, i.e. those phenomena in which the rate of change of a function y is a function of the second power of y, and independent of time except through the changes in y itself. In these second-order growth processes there is sigmoidal growth between two asymptotes; outside these asymptotes there are a hypo- and hyper-critical phase of negative growth.

The fact that growth does not change monotonically, but levels off periodically, is important in the way one visualizes career growth. It is often stated that in the natural sciences all major accomplishments are achieved before age thirty, whereas the arts provide many examples, notably Picasso and Verdi, illustrating that creativity need not be limited by age. It is tempting to look for a cause of this disparity in the context of society's reinforcement of one of the three trifurcation branches resulting from sigmoid superposition. George F. Handel, having achieved success as an opera composer, found that his successful formula, typical of the saturation portion of the sigmoid curve, ultimately failed to attract audiences; when Handel experimented with oratorio, he entered a new sigmoidal growth with phenomenal success. Most artists pass through 'periods' which are distinguished by different media, styles or techniques: although each sigmoidal function is unaffected by external changes, the shift at the trifurcation point appears very much to be influenced by external conditions. It stands to reason that a successful scientist, who is more able than an artist to operate without external communication, would be less motivated to jump to a new sigmoid curve or one of its satellites than the successful artist.

It is inappropriate to discuss in this paper the reinforcement (in behaviorist terms) which society furnishes at the trifurcation points, because I want to avoid value judgements here. The purpose of this paper was not to make such judgements, but rather to provide readers with a conceptual model by which to analyze their own careers and extra-curricular activities, to understand critical moments when decisions are inevitable, and to see historical developments in terms of relatively quiet periods separated by fairly radical transitions.

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